JOURNAL OF ALGEBRA **167**, 104–115 (1994)

The 4-Dimensional Sklyanin Algebra

JOANNA M. STANISZKIS*

Department of Mathematics, University of Washington, Seattle, Washington 98195

Communicated by Susan Montgomery

Received August 28, 1992

In this paper we show that the defining relations and central elements of degree two of the algebra $A(E, \tau)$ defined by E. K. Sklyanin in [4] satisfy simple functional equations. These equations allow us to give shorter proofs of some results in [1, 5, 7]. We also give an explicit construction of most line modules over $A(E, \tau)$ and describe the annihilators of these modules.

1. INTRODUCTION AND DEFINITIONS

Fix $\eta \in \mathbb{C}$ with $\operatorname{Im}(\eta) > 0$, and write $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\eta$.

For each integer $p \geqslant 0$ and each $ab \in \{00, 01, 10, 11\}$ let Θ^p_{ab} denote the space of holomorphic functions on $\mathbb{C}$ such that

$$f(z+1) = e^{-\pi i a} f(z) \qquad \text{and} \qquad f(z+\eta) = e^{-\pi i b - p\pi i(2z+\eta)} f(z)$$

for all $z \in \mathbb{C}$. Such functions are called *theta functions* of *weight* p and *characteristic* ab. We have $\dim \Theta^p_{ab} = p$.

Let $\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}$ be Jacobi's four theta functions associated to Λ, as in Weber's book [10, p. 71]. Thus $\theta_{ab} \in \Theta^1_{ab}$. The only zero of θ_{ab} in the fundamental parallelogram is at $z = (1-b)/2 + ((1+a)/2)\eta$ and for all $z \in \mathbb{C}$ we have $\theta_{ab}(-z) = (-1)^{ab}\,\theta_{ab}(z)$.

Fix $\tau \in \mathbb{C}$ such that τ is not of order 4 in $\mathbb{C}/\Lambda$. Write V for the vector space Θ^4_{00}. Let $\{X_{11}, X_{00}, X_{01}, X_{10}\}$ be the basis of V, where $X_{ab}(z) = \gamma_{ab}\theta_{ab}(\tau)\,\theta_{ab}(2z)$ with $\tau_{11} = \gamma_{00} = i$ and $\gamma_{01} = \gamma_{10} = 1$. Define $E = j_\tau(\mathbb{C}/\Lambda)$, where $j_\tau\colon \mathbb{C}/\Lambda \to \mathbb{P}(V^*) = \mathbb{P}^3$ is given by

$$j_\tau(z) = (X_{11}(z), X_{00}(z), X_{01}(z), X_{10}(z))$$

with respect to the homogeneous coordinates X_{11}, X_{00}, X_{01}, X_{10}. Let (ab, ij, kl) be a cyclic permutation of $(00, 01, 10)$ and define $\alpha_{ab} = (-1)^{a+b}\,[\theta_{11}(\tau)\,\theta_{ab}(\tau)/\theta_{ij}(\tau)\,\theta_{kl}(\tau)]^2$.

* Present address: Department of Mathematics, Texas A&M University, College Station, TX 77843. E-mail: staniszk@math.tamu.edu.

0021-8693/94 $6.00

The 4-dimensional Sklyanin algebra associated to (E, τ) is defined to be the quotient of the tensor algebra

$$A(E, \tau) := T(V)/I,$$

where I is generated by its six dimensional subspace $I_2 \subseteq V \otimes V$ with basis

$$f_{ab} = X_{11} \otimes X_{ab} - X_{ab} \otimes X_{11} - \alpha_{ab}(X_{ij} \otimes X_{kl} + X_{kl} \otimes X_{ij}),$$
$$f'_{ab} = X_{11} \otimes X_{ab} + X_{ab} \otimes X_{11} - X_{ij} \otimes X_{kl} - X_{kl} \otimes X_{ij}.$$

The algebras $A(E, \tau)$ were defined by Sklyanin in [4]. Since E and τ will be fixed throughout the paper, we will just write A for $A(E, \tau)$.

Let $f \in V \otimes V$. Since elements of V are holomorphic functions on $\mathbb{C}$, we will view f as a holomorphic function on $\mathbb{C}^2$ in the obvious way. Furthermore, since with respect to each coordinate f is an element of Θ_{00}^4, the zeros of f can be viewed as points of $\mathbb{C}^2/\Lambda^2 \cong E^2$.

The main result in Section 2 is Theorem 2.2 which gives a criterion for $f \in V \otimes V$ to be an element of I. A consequence of this criterion is an easy proof of the fact discovered by Sklyanin, that the space of meromorphic functions on $\mathbb{C}$ is an A-module. Similarly, in Proposition 2.5 we give a necessary and sufficient condition for $\tilde{\Omega} \in V \otimes V$ to be a preimage of a central element of A.

In Section 3 we apply above results to give an explicit construction of most line modules over A and describe their anihilators. Furthermore, we present shorter proofs of some results in [1].

2. Functional Equations

The defining relations for A satisfy a simple functional equation. The rest of the paper exploits this equation.

Lemma 2.1. *Let $f \in V \otimes V$ and $c \in \mathbb{C}$. If for all $(x, y) \in \mathbb{C}^2$*

$$\frac{f(x, y-\tau)}{\theta_{11}(x-y+2\tau)} + c\frac{f(y, x-\tau)}{\theta_{11}(y-x+2\tau)} = 0 \tag{2.1}$$

then f vanishes on $\Delta_\tau = \{(p, p+\tau) : p \in E\}$.

Proof. Set $y = x + 2\tau$. Then $\theta_{11}(x-y+2\tau) = 0$ and since $4\tau \neq 0$, we have $\theta_{11}(y-x+2\tau) \neq 0$. Therefore we must have cancelation of poles, so $f(x, x+\tau) = 0$. ∎

THEOREM 2.2. *Let $f \in V \otimes V$. Then $f \in I_2$ if and only if for all $(x, y) \in \mathbb{C}^2$*

$$\frac{f(x, y-\tau)}{\theta_{11}(x-y+2\tau)} - \frac{f(y, x-\tau)}{\theta_{11}(y-x+2\tau)} = 0. \tag{2.2}$$

Proof. We will first show that f_{ab} and f'_{ab} satisfy (2.2). Since $\gamma_{ij}\gamma_{kl} = -(-1)^{a+b}\gamma_{11}\gamma_{ab}$, for $(x, y) \in \mathbb{C}^2$ we have

$$\begin{aligned} f_{ab}(x, y-\tau) = \gamma_{11}\gamma_{ab}[&\theta_{11}(\tau)\,\theta_{11}(2x)\,\theta_{ab}(-\tau)\,\theta_{ab}(2y-2\tau) \\ &- \theta_{ab}(\tau)\,\theta_{ab}(2x)\,\theta_{11}(-\tau)\,\theta_{11}(2y-2\tau) \\ &+ (-1)^{a+b}(\theta_{ij}(\tau)\,\theta_{ij}(2x)\,\theta_{kl}(-\tau)\,\theta_{kl}(2y-2\tau) \\ &- \theta_{kl}(\tau)\,\theta_{kl}(2x)\,\theta_{ij}(-\tau)\,\theta_{ij}(2y-2\tau))], \end{aligned}$$

$$\begin{aligned} f'_{ab}(x, y-\tau) = \gamma_{11}\gamma_{ab}\frac{\theta_{11}(\tau)\,\theta_{ab}(\tau)}{\theta_{ij}(\tau)\,\theta_{kl}(\tau)}[&\theta_{ij}(\tau)\,\theta_{11}(2x)\,\theta_{kl}(-\tau)\,\theta_{ab}(2y-2\tau) \\ &- \theta_{kl}(\tau)\,\theta_{ab}(2x)\,\theta_{ij}(-\tau)\,\theta_{11}(2y-2\tau) \\ &+ \theta_{11}(\tau)\,\theta_{ij}(2x)\,\theta_{ab}(-\tau)\,\theta_{kl}(2y-2\tau) \\ &- \theta_{ab}(\tau)\,\theta_{kl}(2x)\,\theta_{11}(-\tau)\,\theta_{ij}(2y-2\tau)]. \end{aligned}$$

To prove that f_{ab} and f'_{ab} satisfy (2.2) we will use Riemann's theta formulas labelled (R_i) in [9, p. 20]. Let $x, y, u, v \in \mathbb{C}$ and $x_1 = \frac{1}{2}(x+y+u+v)$, $y_1 = \frac{1}{2}(x+y-u-v)$, $u_1 = \frac{1}{2}(x-y+u-v)$, and $v_1 = \frac{1}{2}(x-y-u+v)$. The formulas (R_8), (R_{13}), (R_{16}) can be written in the short form as

$$\begin{aligned} &2\theta_{11}(x_1)\,\theta_{11}(y_1)\,\theta_{ab}(u_1)\,\theta_{ab}(v_1) \\ &\quad= \theta_{11}(x)\,\theta_{11}(y)\,\theta_{ab}(u)\,\theta_{ab}(v) - \theta_{ab}(x)\,\theta_{ab}(y)\,\theta_{11}(u)\,\theta_{11}(v) \\ &\qquad+ (-1)^{a+b}[\theta_{ij}(x)\,\theta_{ij}(y)\,\theta_{kl}(u)\,\theta_{kl}(v) - \theta_{kl}(x)\,\theta_{kl}(y)\,\theta_{ij}(u)\,\theta_{ij}(v)]. \end{aligned}$$

Hence for $x = \tau$, $y = 2x$, $u = -\tau$, and $v = 2y - 2\tau$ we have

$$\begin{aligned} f_{ab}(x, y-\tau) = \gamma_{11}\gamma_{ab}2\theta_{11}&(x+y-\tau)\,\theta_{11}(x-y+2\tau) \\ &\times \theta_{11}(x+y-\tau)\,\theta_{11}(y-x). \end{aligned}$$

Similarly, by (R_{19}) with $x = \tau$, $y = 2x$, $u = \tau$, and $v = 2y - 2\tau$, by (R_{20}) with $x = 2x$, $y = -\tau$, $u = \tau$, and $v = 2y - 2\tau$ and by (R_{21}) with $x = -\tau$, $y = \tau$, $u = 2x$, and $v = 2y - 2\tau$ we obtain

$$\begin{aligned} f'_{ab}(x, y-\tau) &= (-1)^a\,2\gamma_{11}\gamma_{ab}\frac{\theta_{11}(\tau)\,\theta_{ab}(\tau)}{\theta_{ij}(\tau)\,\theta_{kl}(\tau)} \\ &\quad\times [\theta_{ij}(x+y-\tau)\,\theta_{ab}(x-y)\,\theta_{11}(y-x-2\tau)\,\theta_{kl}(x+y-\tau)]. \end{aligned}$$

There are similar expressions with x and y interchanged. Using the fact that θ_{11} is odd and the other θ_{ab} are even and cross-multiplying the appropriate equations gives the desired functional equation.

To prove the converse define $\mathscr{S} = \{f \in V \otimes V : f \text{ satisfies } (2.2)\}$. We will show that $\dim \mathscr{S} = \dim I_2$. Let W be the space of theta functions such that for all $z \in \mathbb{C}$

$$g(z+1) = g(z) \qquad \text{and} \qquad g(z+\eta) = e^{-4\pi i(2z+\eta-\tau)} g(z).$$

Write $\Lambda(W \otimes W)$ for the space of skew-symmetric tensors in $W \otimes W$. Since $\dim W = 4$ we have $\dim \Lambda(W \otimes W) = \binom{4}{2}$. To finish the proof we will show that the vector spaces $\mathscr{S}$ and $\Lambda(W \otimes W)$ are isomorphic. Define a linear map $\Upsilon: \mathscr{S} \to \Lambda(W \otimes W)$ by

$$\Upsilon(f)(x, y) = f(x, y-\tau) \frac{\theta_{11}(x-y)}{\theta_{11}(x-y+2\tau)}.$$

By Lemma 2.1 with $c = -1$ we have that $\theta_{11}(x-y+2\tau) = 0$ implies $f(x, y-\tau) = 0$, so $\Upsilon(f)$ is holomorphic. It is easy to check that $\Upsilon(f) \in W \otimes W$. Furthermore, since f satisfies (2.2) and θ_{11} is odd we have $\Upsilon(f)(x, y) = -\Upsilon(f)(y, x)$, so $\Upsilon(f)$ is skew-symmetric. Finally, since Υ is injective and $\dim \mathscr{S} \geq 6$ the result follows. ∎

COROLLARY 2.3. *If $f \in I_2$ then $f(\Delta_\tau) = 0$.*

An easy application of (2.2) is the following result, which is the main step in constructing finite dimensional irreducible representations of A (see [4, 8]).

THEOREM 2.4 [5, Theorem 2]. *Let $\mathscr{M}(\mathbb{C})$ denote the space of meromorphic functions of $\mathbb{C}$. For each $k \in \mathbb{N} \cup \{0\}$, $\mathscr{M}(\mathbb{C})$ is an A-module with the action of $X \in A_1$ on $g \in \mathscr{M}(\mathbb{C})$ given by*

$$(X \cdot g)(z) = \frac{X(z - \frac{1}{2}k\tau)}{\theta_{11}(2z)} g(z+\tau) - \frac{X(-z-\frac{1}{2}k\tau)}{\theta_{11}(2z)} g(z-\tau).$$

Proof. This action makes $\mathscr{M}(\mathbb{C})$ a $T(V)$-module so we must check that if $f \in I_2$ then for all $g \in \mathscr{M}(\mathbb{C})$ $f \cdot g = 0$. Let $X, Y \in V$. Then

$$\begin{aligned}(X \otimes Y \cdot g)(z) = {} & \frac{X(z-\frac{1}{2}k\tau)\, Y(z-\frac{1}{2}k\tau+\tau)}{\theta_{11}(2z)\,\theta_{11}(2z+2\tau)} g(z+2\tau) \\ & + \frac{X(-z-\frac{1}{2}k\tau)\, Y(-z-\frac{1}{2}k\tau+\tau)}{\theta_{11}(2z)\,\theta_{11}(2z-2\tau)} g(z-2\tau)\end{aligned}$$

$$-\left(\frac{X(z-\frac{1}{2}k\tau)\,Y(-z-\frac{1}{2}k\tau-\tau)}{\theta_{11}(2z)\,\theta_{11}(2z+2\tau)}\right.$$
$$\left.-\frac{X(-z-\frac{1}{2}k\tau)\,Y(z-\frac{1}{2}k\tau-\tau)}{\theta_{11}(2z)\,\theta_{11}(-2z+2\tau)}\right) g(z).$$

Let $f \in I_2$ and consider $(f \cdot g)(z)$. The coefficient of $g(z)$ is zero by (2.2) and the coefficients of $g(z+2\tau)$ and $g(z-2\tau)$ are zero since f vanishes on Δ_τ. Hence $\mathscr{M}(\mathbb{C})$ is an A-module. ∎

In [4] Sklyanin found two central elements in A_2, namely,

$$\Omega_1 = -X_{11}^2 + X_{00}^2 + X_{01}^2 + X_{10}^2$$

and

$$\Omega_2 = X_{00}^2 + \left(\frac{1+\alpha_{00}}{1-\alpha_{01}}\right) X_{01}^2 + \left(\frac{1-\alpha_{00}}{1+\alpha_{10}}\right) X_{10}^2.$$

Define the following preimages of Ω_1 and Ω_2 in $V \otimes V$:

$$\tilde{\Omega}_1 = -X_{11} \otimes X_{11} + X_{00} \otimes X_{00} + X_{01} \otimes X_{01} + X_{10} \otimes X_{10},$$

$$\tilde{\Omega}_2 = X_{00} \otimes X_{00} + \left(\frac{1+\alpha_{00}}{1-\alpha_{01}}\right) X_{01} \otimes X_{01} + \left(\frac{1-\alpha_{00}}{1+\alpha_{10}}\right) X_{10} \otimes X_{10}.$$

The next result shows that elements of $\mathbb{C}\tilde{\Omega}_1 \oplus \mathbb{C}\tilde{\Omega}_2$ satisfy a simple functional equation. This equation will allow as to give short proof of Theorem 3.2.

PROPOSITION 2.5. *Let $\tilde{\Omega} \in V \otimes V$. Then $\tilde{\Omega} \in \mathbb{C}\tilde{\Omega}_1 \oplus \mathbb{C}\tilde{\Omega}_2$ if and only if for all $(x, y) \in \mathbb{C}^2$*

$$\frac{\tilde{\Omega}(x, y-\tau)}{\theta_{11}(x-y+2\tau)} + \frac{\tilde{\Omega}(y, x-\tau)}{\theta_{11}(y-x+2\tau)} = 0. \tag{2.3}$$

Proof. We will first show that $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ satisfy (2.3). By [9, p. 22, Addition Formulas (A_4), (A_6), (A_8), (A_9)] $\tilde{\Omega}_2$ can be rewritten as

$$\Omega_2 = X_{00} \otimes X_{00} + \frac{\theta_{01}(2\tau)\,\theta_{01}(0)\,\theta_{00}(\tau)^2}{\theta_{00}(2\tau)\,\theta_{00}(0)\,\theta_{01}(\tau)^2} X_{01} \otimes X_{01}$$
$$+ \frac{\theta_{10}(2\tau)\,\theta_{10}(0)\,\theta_{00}(\tau)^2}{\theta_{00}(2\tau)\,\theta_{00}(0)\,\theta_{10}(\tau)^2} X_{10} \otimes X_{10}.$$

We will use the Riemann's theta formula labelled (R_5) in [9, p. 20], which says that for all $x, y, u, v \in \mathbb{C}$

$$\begin{aligned}
&2\theta_{11}(x_1)\,\theta_{11}(y_1)\,\theta_{11}(u_1)\,\theta_{11}(v_1)\\
&\quad=\theta_{00}(x)\,\theta_{00}(y)\,\theta_{00}(u)\,\theta_{00}(v)-\theta_{01}(x)\,\theta_{01}(y)\,\theta_{01}(u)\,\theta_{01}(v)\\
&\qquad-\theta_{10}(x)\,\theta_{10}(y)\,\theta_{10}(u)\,\theta_{kl}(v)+\theta_{11}(x)\,\theta_{11}(y)\,\theta_{11}(u)\,\theta_{11}(v),
\end{aligned}$$

where $x_1=\frac{1}{2}(x+y+u+v)$, $y_1=\frac{1}{2}(x+y-u-v)$, $u_1=\frac{1}{2}(x-y+u-v)$, and $v_1=\frac{1}{2}(x-y-u+v)$. Since $\theta_{11}(0)=0$, we have

$$\begin{aligned}
\tilde{\Omega}_1(x, y-\tau)&=\sum_{kl}\gamma_{kl}^2(\theta_{kl}(\tau)\,\theta_{kl}(-\tau)\,\theta_{kl}(2x)\,\theta_{kl}(2y-2\tau)\\
&=2\theta_{11}(x+y-\tau)^2\,\theta_{11}(x-y+2\tau)\,\theta_{11}(x-y), \qquad (2.4)\\
\tilde{\Omega}_2(x, y-\tau)&=\frac{\theta_{00}(\tau)^2}{\theta_{00}(0)\,\theta_{00}(2\tau)}\sum_{kl}\gamma_{kl}^2\theta_{kl}(0)\,\theta_{kl}(2\tau)\,\theta_{kl}(2x)\,\theta_{kl}(2y-2\tau)\\
&=2\frac{\theta_{00}(\tau)^2}{\theta_{00}(0)\,\theta_{00}(2\tau)}\theta_{11}(x+y)\,\theta_{11}(x+y-2\tau)\\
&\qquad\times\theta_{11}(x-y)\,\theta_{11}(x-y+2\tau). \qquad (2.5)
\end{aligned}$$

We obtain similar expressions with x and y interchanged. Using the fact that θ_{11} is odd and cross-multiplying the appropriate equations gives the desired functional equation.

To prove the converse we need to show that the space of elements in $V\otimes V$ which satisfy (2.3) is 2-dimensional. This follows from Lemma 2.1 with $c=1$ and [7, 3.4, 3.9]. ∎

COROLLARY 2.6. *If $\tilde{\Omega}\in\mathbb{C}\tilde{\Omega}_1\oplus\mathbb{C}\tilde{\Omega}_2$ then $\tilde{\Omega}(\Delta_\tau)=0$.*

Remarks. (a) A *point module* is a cyclic, graded A-module with Hilbert series $(1-t)^{-1}$. By Corollaries 2.3, 2.6 we have that for each $p\in E$ there is a point module $M(p)$ for the algebra $A/\langle\Omega_1,\Omega_2\rangle$. Using (2.2) and the geometry of E in $\mathbb{P}^3$ it can be shown that the point modules for A are perametrized by points of $E\cup\{4 \text{ points}\}$ (see [6] for details).

(b) Let $A_n(E,\tau)=T(V)/\langle R_A\rangle$ with $R_A\subset V\otimes V$ be the n-dimensional Sklyanin algebra defined by A. V. Odesskii and B. L. Feigin in [3, Sect. 2]. Let $\{X_k:=\theta_k : k\in\mathbb{Z}/n\mathbb{Z}\}$ be the basis of V (see [2, 3] for details). Write θ for a theta function satisfying

$$\theta(z+1)=\theta(z) \qquad\text{and}\qquad \theta(z+\eta)=-e^{-2\pi iz}\theta(z).$$

Let $f \in V \otimes V$. Then $f \in R_A$ if and only if

$$\frac{f(x, y-2\tau)}{\theta(x-y)\,\theta(y-x-n\tau)} + \frac{f(y, x-2\tau)}{\theta(y-x)\,\theta(x-y-n\tau)} = 0 \tag{2.6}$$

for all $x, y \in \mathbb{C}$. The idea for (2.6) was first suggested by the basis of R_A given in [3, Sect. 2].

3. Line Modules

A *line module* is a cyclic graded A-module with Hilbert series $(1-t)^{-2}$. It is proved in [1] that line modules are in bijection with secant lines of E. Let $l_{pq} \subseteq \mathbb{P}(V^*) = \mathbb{P}^3$ be the secant line passing through points $p, q \in E$. If $u, v \in A_1$ are such that $\mathcal{V}(u, v) = l_{pq}$, then $A/Au + Av$ is a line module denoted by $M(p, q)$. Proposition 3.1, which should be compared with [2, 3.2; 1, 5.6], gives an explicit construction of $M(p, q)$ for most (p, q).

Proposition 3.1. *Fix $p, q \in E$ such that $p - q \notin \mathbb{Z} \cdot 2\tau$. Then $M(p, q)$ has basis $\{e_{ij} : (i, j) \in \mathbb{N}^2\}$ with the property that if $X \in A_1$ then*

$$X \cdot e_j = \frac{X(q+(i-j)\tau)}{\theta_{11}(p-q-2(i-j)\tau)} e_{i,j+1} - \frac{X(p+(j-i)\tau)}{\theta_{11}(p-q-2(i-j)\tau)} e_{i+1,j}.$$

Proof. Let $p, q \in \mathbb{C}$ and define a graded $T(V)$-module $M = \bigoplus M_n$, where $M_n = \bigoplus_{i+j=n} \mathbb{C} e_{ij}$, with above action. We will show that it is also an A-module and that up to isomorphism the module $M(p, q)$ depends only on the image of p, q in $\mathbb{C}/\Lambda$. First note that since $p - q \notin \mathbb{Z} \cdot 2\tau$, $\theta_{11}(p-q-2(i-j)\tau) \neq 0$ for all $(i, j) \in \mathbb{N}^2$, so the formula for $X \cdot e_{ij}$ makes sense. Let $X, Y \in V$. Then

$$\begin{aligned}
X \otimes Y \cdot e_{ij} &= X \cdot \left(\frac{Y(q+(i-j)\tau)}{\theta_{11}(p-q-2(i-j)\tau)} e_{i,j+1} + \right. \\
&\quad \left. + \frac{Y(p+(j-i)\tau)}{\theta_{11}(p-q-2(i-j)\tau)} e_{i+1,j} \right) \\
&= \frac{X(q+(i-j-1)\tau)\, Y(q+(i-j)\tau)}{\theta_{11}(p-q-2(i-j-1)\tau)\,\theta_{11}(p-q-2(i-j)\tau)} e_{i,j+2} \\
&\quad + \frac{X(p+(j-i-1)\tau)\, Y(p+(j-i)\tau)}{\theta_{11}(p-q-2(i-j+1)\tau)\,\theta_{11}(p-q-2(i-j)\tau)} e_{i+2,j} \\
&\quad - \left(\frac{X(p+(j+1-i)\tau)\, Y(q+(i-j)\tau)}{\theta_{11}(p-q-2(i-j-1)\tau)\,\theta_{11}(p-q-2(i-j)\tau)} \right. \\
&\quad \left. + \frac{X(q+(i+1-j)\tau)\, Y(p+(j-i)\tau)}{\theta_{11}(p-q-2(i-j+1)\tau)\,\theta_{11}(p-q-2(i-j)\tau)} \right) e_{i+1,j+1}.
\end{aligned}$$

Hence if $f \in V \otimes V$ then

$$
\begin{aligned}
f \cdot e_{ij} = {} & \frac{f(y-2\tau, y-\tau)}{\theta_{11}(x-y+2\tau)\,\theta_{11}(x-y)} e_{i,j+2} \\
& + \frac{f(x-2\tau, x-\tau)}{\theta_{11}(x-y-2\tau)\,\theta_{11}(x+y)} e_{i+2,j} \\
& - \left(\frac{f(x, y-\tau)}{\theta_{11}(x-y+2\tau)\,\theta_{11}(x-y)} \right. \\
& \left. - \frac{f(y, x-\tau)}{\theta_{11}(y-x+2\tau)\,\theta_{11}(x-y)} \right) e_{i+1,j+1},
\end{aligned} \tag{3.1}
$$

where $x = p + (j+1-i)\tau$ and $y = q + (i+1-j)\tau$. Let $f \in I_2$ and consider $f \cdot e_{ij}$. The coefficient of $e_{i+1,j+1}$ is zero by (2.2) and the coefficients of $e_{i+2,j}$ and $e_{i,j+2}$ are zero since f vanishes on Δ_τ. Hence M is a graded A-module.

It remains to show that M is generated by M_0. For $(i, j) \in \mathbb{N}^2$ define the set $\mathcal{T}_{ij} = \{p + (j-i)\tau, q + (i-j)\tau\}$. By the hypothesis on $p - q$ each $\mathcal{T}_{ij}$ has cardinality 2. Since each element of A_1 is a theta function of weight 4, it is determined up to a scalar multiple by its four zeroes [10]. Hence for each (i, j) there exists $X \in A_1$ such that X vanishes on one element of $\mathcal{T}_{ij}$ and is nonzero on the other. Thus for all (i, j) there exist $X, Y \in A_1$ such that $X \cdot e_{ij} = e_{i+1,j}$ and $Y \cdot e_{ij} = e_{i,j+1}$. By induction it follows that M is geneated by M_0. $H_M(t) = (1-t)^{-2}$ and $\mathcal{V}(\mathrm{Ann}_{A_1} e_{00}) = l_{pq}$, M is a line module isomorphic to $M(p, q)$.

To finish the proof we have to show that up to isomorphism the module $M(p, q)$ depends only on the image of p, q in $\mathbb{C}/\Lambda$. Let $p' \in p + \Lambda$ and $q' \in q + \Lambda$. Let $\{v_{ij} : (i, j) \in \mathbb{N}^2\}$ be a basis for $M(p', q')$ such that

$$
X * v_{ij} = \frac{X(q' + (i-j)\tau)}{\theta_{11}(p' - q' - 2(i-j)\tau)} v_{i,j+1} - \frac{X(p' + (j-i)\tau)}{\theta_{11}(p' - q' - 2(i-j)\tau)} v_{i+1,j}.
$$

If $p' = p + m$ and $q' = q$ for $m \in \mathbb{Z}$, then clearly $M(p, q) \cong M(p', q')$ via $e_{ij} \mapsto (-1)^{m(i+j)} v_{ij}$. Now let $p' = p + \eta$ and $q' = q$. Set

$$
\alpha_1(i, j) = e^{-\pi i(6p + 2q + 4(j-i)\tau + 3\eta)} \quad \text{and} \quad \alpha_2(i, j) = -e^{\pi i(2(p-q) + 4(j-i)\tau + \eta)}.
$$

Define the linear map Φ: $M(p, q) \to M(p', q')$ by $\Phi(e_{ij}) = C_{ij} v_{ij}$, where the scalars $C_{ij} \in \mathbb{C}$ are defined inductively by

$$
C_{00} = 1, \qquad C_{i+1,j} = \alpha_1(i, j)\, C_{ij}, \qquad C_{i,j+1} = \alpha_2(i, j)\, C_{ij}.
$$

An easy computation shows that Φ is an A-module isomorphism. ∎

THEOREM 3.2. *Let $p, q \in E$. If $p+q=z$ then $M(p, q)$ is annihilated by*

$$\Omega = \frac{\theta_{00}(\tau)^2}{\theta_{00}(0)\,\theta_{00}(2\tau)}\,\theta_{11}(z)\,\theta_{11}(z+2\tau)\,\Omega_1 - \theta_{11}(z+\tau)^2\,\Omega_2.$$

Proof. View $M(p, q)$ as a $T(V)$-module. Pick $p, q \in E$ such that $p+q=z$ and $p-q \notin \mathbb{Z}\cdot 2\tau$. Let $\{e_{ij}\}$ be the basis of $M(p, q)$ as in Proposition 3.1. Write $\tilde{\Omega}$ for the preimage of Ω in $\mathbb{C}\tilde{\Omega}_1 \oplus \mathbb{C}\tilde{\Omega}_2$. Since $\tilde{\Omega}$ vanishes on Δ_τ, (3.1) and (2.3) yield

$$\tilde{\Omega}\cdot e_{00} = -\frac{2\tilde{\Omega}(p+\tau, q)}{\theta_{11}(p-q+2r)\,\theta_{11}(p-q)}\,e_{11}.$$

Applying (2.4) and (2.5) with $x=p+\tau$ and $y=q+\tau$ gives

$$\tilde{\Omega}_1(p+\tau, q) = 2\theta_{11}(p+q+\tau)^2\,\theta_{11}(p-q+2\tau)\,\theta_{11}(p-q),$$

$$\tilde{\Omega}_2(p+\tau, q) = \frac{2\theta_{00}(\tau)^2}{\theta_{00}(0)\,\theta_{00}(2\tau)}\,\theta_{11}(p+q+2\tau)\,\theta_{11}(p+q) \times \theta_{11}(p-q)\,\theta_{11}(p-q+2\tau).$$

Hence

$$\begin{aligned}\tilde{\Omega}(p+\tau, q) &= \frac{\theta_{00}(\tau)^2}{\theta_{00}(0)\,\theta_{00}(2\tau)}\,\theta_{11}(p+q)\,\theta_{11}(p+q+2\tau)\,\tilde{\Omega}_1(p+\tau, q)\\ &\quad - \theta_{11}(p+q+\tau)^2\,\tilde{\Omega}_2(p+\tau, q)\\ &= 0.\end{aligned}$$

Therefore $\tilde{\Omega}\cdot e_{00}=0$ and so $\Omega\cdot e_{00}=0$ in the A-module $M(p, q)$. Since Ω is central, $\Omega\cdot M(p, q)=0$. Finally, by [1, 6.6] Ω annihilates all line modules $M(p, q)$ with $p+q=z$. ∎

Remark. Write $\Omega(z)$ for Ω in Theorem 3.2. In this notation $\Omega_1 = \Omega(-\tau)$ and $\Omega_2 = \Omega(0)$.

Let $\mathrm{Aut}(A)$ denote the group of automorphisms of A which preserve the grading. Fix an A-module M. For $\varphi \in \mathrm{Aut}(A)$ define M^φ, the *twist* of M to be the A-module which is M as a $\mathbb{C}$-vector space with A-action given by $a * m = \varphi(a)\cdot m$ for all $a \in A$ and for all $m \in M$. If $\lambda \in \mathbb{C}^\times$, then write φ_λ for the automorphism of A given by $\varphi_\lambda(a) = \lambda^n a$ for all $a \in A_n$. Hence we consider $\mathbb{C}^\times$ as a subgroup of $\mathrm{Aut}(A)$. By [8, Sect. 2] there is a short exact sequence

$$1 \to \mathbb{C}^\times \to \mathrm{Aut}(A) \to E_4 \to 1$$

and a group homomorphism $\Phi\colon E_4 \to \mathrm{Aut}(A)/\mathbb{C}^\times$, where E_4 denotes the 4-torsion subgroup of E. If M is a graded module, then $M^{\varphi_\lambda} \cong M$ for all

$\lambda \in \mathbb{C}^\times$, so it is the action of $\mathrm{Aut}(A)/\mathbb{C}^\times$ on the graded A-modules which is important.

COROLLARY 3.3. *Let* $p, q \in E$ *and* $\xi = (1-b)/4 + ((1-a)/4)\eta \in E_4 \setminus \{0\}$, *where* $ab \in \{00, 01, 10\}$. *Write* $Z_2 = \mathbb{C}\Omega_1 \oplus \mathbb{C}\Omega_2$. *Then*

(a) $\mathrm{Ann}_{Z_2} M(p, q) = \mathbb{C}\Omega(p+q)$.

(b) $\mathrm{Ann}_{Z_2} M(p, q)^{\Phi(\xi)} = \mathbb{C}((\theta_{00}(\tau)^2/\theta_{00}(0)\ \theta_{00}(2\tau))\ \theta_{ab}(p+q)\ \theta_{ab}(p+q+2\tau)\ \Omega_1 - \theta_{ab}(p+q+\tau)^2\,\Omega_2)$.

Proof. Part (a) is a consequence of Theorem 3.2 and [1, 6.1] which says that $\mathrm{Ann}_{Z_2} M(p, q)$ is 1-dimensional.

(b) By [8, 2.3(c)] $M(p, q)^{\Phi(\xi)} \cong M(p+\xi, q+\xi)$. Hence by part (a)

$$\begin{aligned}\mathrm{Ann}_{Z_2} M(p, q)^{\Phi(\xi)} &= \mathbb{C}\Omega(p+q+2\xi)\\ &= \mathbb{C}\left(\frac{\theta_{00}(\tau)^2}{\theta_{00}(0)\,\theta_{00}(2\tau)}\right.\\ &\qquad \times \theta_{11}(p+q+2\xi)\,\theta_{11}(p+q+2\tau+2\xi)\Omega_1\\ &\qquad \left. - \theta_{11}(p+q+\tau+2\xi)^2\,\Omega_2\right)\end{aligned}$$

By [10, (8), p. 73] $\theta_{11}(z+(1-b)/4+((1-a)/4)\eta = e^{\pi i[b/2-(1-a)(\eta/4+z)]}\theta_{ab}(z)$. Hence

$$\begin{aligned}&\Omega(p+q+2\xi)\\ &\quad = e^{\pi i[b-(1-a)(\eta/2+2(p+q+\tau))]}\left(\frac{\theta_{00}(\tau)^2}{\theta_{00}(0)\,\theta_{00}(2\tau)}\theta_{ab}(p+q)\right.\\ &\qquad \left.\times\,\theta_{ab}(p+q+2\tau)\Omega_1 - \theta_{ab}(p+q+\tau)^2\,\Omega_2\right)\end{aligned}$$

and the result follows. ∎

Remark. When τ is of infinite order in E, then by [1, 6.3] the annihilator of a line module $M(p, q)$ is generated by $\Omega(p+q)$.

Theorem 3.2 allows us to give shorter proofs of some results in [1].

PROPOSITION 3.4 [1, 6.9]. *Let* $z, z' \in E$. *Then* $\Omega(z) = \Omega(z')$ *(up to a scalar multiple) if and only if* $z = z'$ *or* $z + z' + 2\tau = 0$.

Proof. By Theorem 3.2 we have $\Omega(z) = \Omega(z')$ if and only if

$$\theta_{11}(z)\,\theta_{11}(z+2\tau)\,\theta_{11}(z'+\tau)^2 = \theta_{11}(z')\,\theta_{11}(z'+2\tau)\,\theta_{11}(z+\tau)^2. \quad (3.2)$$

The Addition Formula labeled (4) in [10, p. 77] says that for all $u, v \in \mathbb{C}$

$$\theta_{01}(0)^2\,\theta_{11}(u+v)\theta_{11}(u-v)=\theta_{11}(u)^2\,\theta_{01}(v)^2-\theta_{01}(u)^2\,\theta_{11}(v)^2.$$

Thus (4) applied twice hields that (3.2) is equivalent to

$$\begin{aligned}0&=\theta_{11}(z+\tau)^2\,\theta_{01}(z'+\tau)^2-\theta_{11}(z'+\tau)^2\,\theta_{01}(z+\tau)^2\\&=\theta_{01}(0)^2\,\theta_{11}(z+z'+2\tau)\,\theta_{11}(z-z')\end{aligned}$$

Hence the result. ∎

PROPOSITION 3.5 [1, 6.13]. *Every element $\Omega \in \mathbb{C}\Omega_1 \oplus \mathbb{C}\Omega_2$ annihilates some line module.*

Proof. By Theorem 3.2 it is enough to show that every element of $\mathbb{P}(\mathbb{C}\Omega_1 \oplus \mathbb{C}\Omega_2) \cong \mathbb{P}^1$ is of the form $\Omega(z)$ for some $z \in E$. Define a map $F: \mathbb{C} \to \mathbb{P}(\mathbb{C}\Omega_1 \oplus \mathbb{C}\Omega_2)$ by

$$F(u)=\frac{\theta_{00}(\tau)^2}{\theta_{00}(0)\,\theta_{00}(2\tau)}\,\frac{\theta_{11}(u)\,\theta_{11}(u+2\tau)}{\theta_{11}(u+\tau)^2}.$$

Since $F(u+1)=F(u)$ and $F(u+\eta)=F(u)$, $F: E \to \mathbb{P}^1$ is a nonconstant holomorphic mapping of Riemann surfaces. Since E is compact, F is surjective. Hence the result. ∎

ACKNOWLEDGMENT

I thank S. P. Smith for introducing me to Sklyanin algebras and for numerous interesting discussions.

REFERENCES

1. T. LEVASSEUR AND S. P. SMITH, Modules over the 4-dimensional Sklyanin algebra, *Bull. Soc. Math. France* **121** (1993), 35–90.
2. A. V. ODESSKII AND B. L. FEIGIN, Sklyanin algebras associated with an elliptic curve, preprint, 1989. [Russian]
3. A. V. ODESSKII AND B. L. FEIGIN, Elliptic Sklyanin algebras (in Russian), *Funktsional Anal. i Prilozhen* **23**, No. 3 (1989), 45–54.
4. E. K. SKLYANIN, Some algebraic structures connected to the Yang–Baxter equation, *Functional Anal. Appl.* **16** (1982), 27–34.
5. E. K. SKLYANIN, Some algebraic structures connected to the Yang–Baxter equation. Representations of Quantum algebras, *Functional Anal. Appl.* **17** (1983), 273–284.
6. S. P. SMITH, Point modules over Sklyanin algebras, *Math. Zeit.*, to appear.

7. S. P. Smith and J. T. Stafford, Regularity of the 4-dimensional Sklyanin algebra, *Compositio Math.* **83** (1992), 259–289.
8. S. P. Smith and J. M. Staniszkis, Irreducible representations of the 4-dimensional Sklyanin algebra at points of infinite order, *J. Algebra* **160** (1993), 57–86.
9. D. Mumford, Tata lectures on Theta I, *in* "Progress in Math," Birkhäuser, Boston, 1984.
10. H. M. Weber, "Elliptische Functionen und Algebraische Zahlen. Lehrbuch der Algebra," Vieweg und Sohn, Braunschweig, 1908.